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Progressive cross waves due to the horizontal oscillations of a vertical cylinder in water. Evolution equations

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Abstract. The paper deals with the theoretical analysis of progressive cross waves excited due to the horizontal oscillations of a vertical, surface-piercing circular cylinder in water of constant depth. Although cross waves are a phenomenon well known in laboratory wave tanks, it seems that they have not been observed around horizontally oscillating structures in fluid up to now. Such observations have recently been carried out by the authors on various models of offshore gravity platforms subjected to earthquake-like horizontal excitation in a water tank. The theoretical analysis of the problem is based on a method developed by Becker and Miles (1992) for the radial cross waves due to the motion of an axisymmetric cylindrical wavemaker. Whitham's average-Lagrangian approach is applied. It is shown that the energy transfer to the cross wave is described by the functional which is quadratic, both in the forced basic wave and in the cross wave. Therefore, the solution to second-order problems is necessary for the derivation of the evolution equations. The evolution of the cross wave is found to be described by two complex nonlinear partial differential equations with coefficients depending on a slow radial variable both in linear and nonlinear terms. The evolution equations are coupled through the nonlinear terms and through the boundary conditions as well.

Keywords: cross waves, evolution equations, nonlinear wave radiation, average-Lagrangian method.

1. Introduction

Cross waves are a phenomenon very well known in laboratory wave tanks. They have been observed during experiments both with generic wavemakers and vertically oscillating, partially immersed structures, as well as in horizontally or vertically vibrating water tanks (see *e.g.* [1, 2]).

The first observation of cross waves can be traced to Faraday (see [3]), who carried out experiments with a vibrating plate and a cork. One hundred years later, a similar discovery was made by Schuler [4] during experiments with a vertically oscillating sphere, a plate and with a wedge. The cross waves in a rectangular channel have been investigated experimentally by Barnard and Pritchard [5] and more recently by Underhill *et al.* [6]. The quantitative experimental study of the cross waves due to vertically oscillating, half-submerged spheres has been carried out by Tatsuno *et al.* [7]. Taneda [8] observed during similar experiments the transition from the outwardly propagating concentric waves to the radially decaying cross waves.

Recently, stable cross waves have been observed by the authors during experiments with various models of offshore gravity platforms subjected to earthquake-like horizontal excitation in a water tank. This observation stimulated a whole series of large-scale experiments with

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different structure models (circular and rectangular cylinders, monotower-type and smooth axisymmetric structures, multiple vertical cylinders) mounted on the shaker plate in the bottom of the water tank and driven harmonically in the horizontal direction. The results of the measurements will soon be published separately; here we mention only that, as soon as the excitation amplitude reached some limiting value, stable progressive cross waves were induced for any excitation frequency and, what is more important, for any type of the structure used. To the authors' knowledge, this is the first time that three-dimensional cross waves due to the horizontal oscillation of free-surface piercing structures have been observed.

The spectral analysis of the measured pressure signals revealed the harmonic components of $\frac{1}{2}\omega$, ω, $\frac{3}{2}\omega$ and 2ω, ω being the excitation frequency. It is a fundamental feature of the phenomenon confirmed in all earlier experimental works that, whereas the directly forced wave (either plane or three-dimensional) has the same frequency as the wavemaker, the cross wave has half that frequency. The problem can then be interpreted in the context of the parametric resonance, in which energy is transferred from the forced wave to the cross wave through nonlinear interactions. Thus, the parasitical, in the context of the laboratory wavemakers, cross waves should now be viewed as the parametrically excited instability of a three-dimensional radiation problem.

The generation of cross waves has not only been studied experimentally, but also theoretically. The first theoretical analysis was given by Garrett [9] who studied the standing cross waves in a short tank for a symmetrical (with respect to the vertical mid-plane of the channel) wavemaker. He linearized the boundary condition at the wavemaker and the boundary conditions at the free surface and obtained, after spatial averaging, Mathieu's equation for the amplitude of the cross wave. Later, Mahony [10] and Jones [11] studied the same problem, but on the assumption of progressive waves in a long channel. Mahony, similarly to Garrett, linearized the boundary conditions at the wavemaker, whereas Jones carried out the nonlinear analysis, using a perturbation method up to the third order of accuracy. Jones obtained the evolution equations for the components of the complex, slowly varying in time and space amplitude of the cross wave with the use of the resonance equations for third-order wave components. These equations could then be combined to obtain a cubic Schrödinger equation in a semi-infinite domain.

A completely different approach based on Whitham's average-Lagrangian method was proposed for the analysis of cross waves by Miles, and Becker and Miles in a series of papers [12, 13, 14, 15]. Using the variational formulation, they were able to avoid many of the complications of a perturbation method and could not only analyse the cross waves in a rectangular channel, but also the radial cross waves due to an axisymmetric wavemaker. The latter problem was thought to be an asymptotic approximation to Faraday's experiment with a vertically oscillating sphere.

Their last paper [15] is of particular importance for the problem considered in the present work. They derive an evolution equation for a progressive radial cross wave excited by a cylindrical wavemaker with the prescribed, radial displacement

$$r = r_1 + \chi(z, t), \qquad \chi = a\, f(kz) \sin 2\omega t. \tag{1}$$

Assuming that the amplitude of the cross wave varies slowly in time and in space, they obtain an evolution equation that differs from the cubic Schrödinger equation only in the presence of a factor $1/R$ in the nonlinear term, where R is a slow radial variable. Then, they incorporate weak, linear damping and obtain the transition conditions at which the forced concentric wave loses stability to a parametrically forced cross wave.

At first glance the problem considered in the present work looks very similar to that considered by Becker and Miles in [15]. However, there are some crucial differences which lead to qualitatively new results.

Firstly, Becker and Miles consider a purely axisymmetric problem (with respect to the directly forced wave), whereas only the wavemaker (circular cylinder) is axisymmetric in the present work. In consequence, a much more complicated form of the cross wave, with two slowly-varying amplitudes, is required.

Secondly, they carry out the analysis on the implicit assumption that the radial displacement of the wavemaker vanishes as $z \to -\infty$. This enables them to solve the problem, using a deep-water approximation. Unfortunately, this is not the case in the present work, since the excitation does not depend on the z-coordinate. Therefore, strictly speaking, the deep-water approximation can only be used for the cross-wave solution which satisfies a homogeneous boundary condition on the wavemaker.

Finally, in the work of Becker and Miles, the energy transfer to the cross-wave is described by a functional which is linear in the forced wave and quadratic in the cross wave. Thus, they need to retain terms of, at the most, second-order in their functional. Subsequently, only the first-order waves are necessary for the derivation of the evolution equation. In contrast to that, the exitation depends on the azimuthal coordinate ϑ in the problem considered here and energy is transfered through higher-order (quartic) interactions. The functional is quadratic both in the forced wave and in the cross wave and comprises terms up to fourth order. Hence, the solution to second-order problems is also necessary for the derivation of the evolution equations.

In the following sections, a variational formulation of the problem is given. Then, the trial solution and the governing equations for its components are developed. The required solution to the first- and second-order problems follows in the next section. Further, the averaged Lagrangian is calculated, and finally, the evolution equations, together with appropriate boundary conditions for complex slowly-varying amplitudes of the cross wave, are derived from Hamilton's principle. Since the theoretical results will be compared in the future with results of large-scale experiments, the effects of surface tension are neglected in the present work.

2. Mathematical formulation

Consider a surface-piercing, circular cylinder founded on the bottom in water of constant depth h. The origin of a fixed coordinate system is located at the undisturbed free surface and the vertical z-axis is positive upward (see Figure 1).

The forced oscillations of the cylinder axis are described by the following displacement function

$$u(t) = u_0 \sin 2\omega t \quad \text{for} \quad t \geqslant 0 \tag{2}$$

in the direction of the x-coordinate. Assuming that $u_0 < r_1$, we can describe the instantaneous cylinder surface (see Figure 2) in cylindrical coordinates (r, ϑ, z) as

$$r = f(\vartheta, t) = u(t)\cos\vartheta + \sqrt{r_1^2 - u^2(t)\sin^2\vartheta} = r_1 + \xi(\vartheta, t). \tag{3}$$

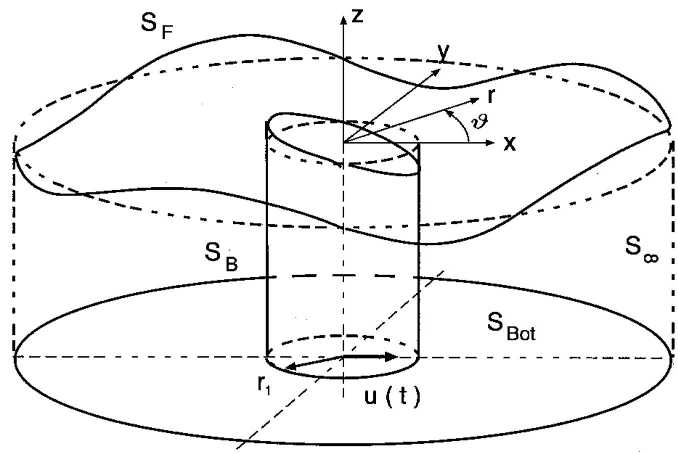

Figure 1. Definition sketch.

Under the assumptions that the flow is irrotational and the fluid incompressible, there exists a velocity potential ϕ describing the waves radiated by the cylinder. The governing equations in cylindrical coordinates for ϕ and the wave elevation η are

$$\nabla^2\phi = 0 \quad (r_1 + \xi < r < \infty, \;\; 0 \leqslant \vartheta < 2\pi, \;\; -h < z < \eta), \tag{4}$$

$$\phi_{,z} = \eta_{,t} + \nabla\phi \cdot \nabla\eta \quad (z = \eta), \tag{5}$$

$$\phi_{,t} + \tfrac{1}{2}(\nabla\phi)^2 + g\eta = 0 \quad (z = \eta), \tag{6}$$

$$\phi_{,r} = \xi_{,t} + \nabla\phi \cdot \nabla f \quad (r = f = r_1 + \xi) \tag{7}$$

$$\phi_{,z} = 0 \quad (z = -h), \tag{8}$$

together with an appropriate radiation condition and the requirement that ϕ and η be periodic in ϑ. Partial derivatives are denoted by ( ),.

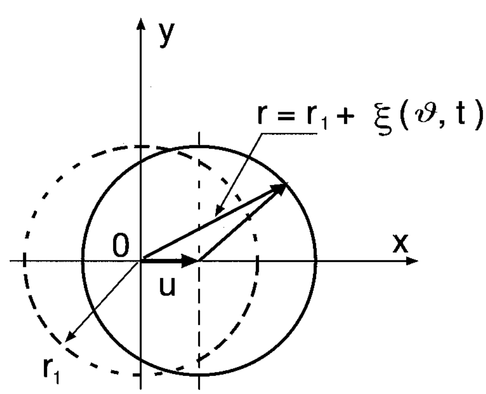

Figure 2. Description of the instantaneous cylinder surface.

This boundary value problem follows from Hamilton's principle in the form

$$\delta \int_{t_1}^{t_2} \hat{L}\,\mathrm{d}t = 0, \tag{9}$$

where the Lagrangian $\hat{L}$ has been derived by Luke [16]

$$\hat{L} = -\int_V [\phi_{,t} + \tfrac{1}{2}(\nabla\phi)^2 + gz]\,\mathrm{d}V, \tag{10}$$

and the volume integral is over the domain bounded by the cylinder ($r = r_1 + \xi$), the free surface ($z = \eta$) and the bottom ($z = -h$).

An equivalent form of the Lagrangian, which is more convenient for the analysis of cross waves, has been derived by Becker and Miles [14]. Following their derivation, we obtain

$$\begin{aligned} 2L \;=\; \int_0^{2\pi} \mathrm{d}\vartheta \left\{ \int_{-h}^{z_0}\int_{r_0}^{\infty} \phi\nabla^2\phi r\,\mathrm{d}r\,\mathrm{d}z + \int_{r_0}^{\infty} [\phi(2\eta_{,t} - \phi_{,z} + \nabla\phi\nabla\eta) - g\eta^2]_{z=\eta} r\,\mathrm{d}r \right. \\ \left. + \int_{-h}^{z_0} [\phi(\phi_{,r} - \nabla\phi\nabla f - 2\xi_{,t})]_{r=f} f(\vartheta, t)\,\mathrm{d}z \right\}, \end{aligned} \tag{11}$$

where $r_0(\vartheta, t)$ and $z_0(\vartheta, t)$ are the coordinates of the intersection of the instantaneous cylinder surface with the free surface.

The boundary value problem (4–8) admits a directly forced wave solution with frequency 2ω which is stable for sufficiently small u_0, but as u_0 is increased it may lose stability to a radial cross wave. The cross wave is described by the same boundary-value problem with a homogeneous boundary condition on the wetted cylinder surface.

3. Trial solution and governing equations

In the analysis which follows the variables are made dimensionless by the relations

$$(\tilde{r},\ \tilde{r}_1,\ \tilde{z},\ \tilde{h}) = k\,(r,\ r_1,\ z,\ h), \qquad \theta = \omega t \tag{12}$$

with subsequent omitting of tildas, and a small parameter $\varepsilon = ku_0 \ll 1$ is defined with k being the wavenumber of the cross wave. The problem will be solved under the assumption that the nonlinearity can transfer energy from the forced wave to the cross wave if the exitation frequency 2ω is approximately twice one of the natural frequencies ω_k of the cross wave according to

$$\omega^2 - \omega_k^2 = O(\varepsilon^4\omega^2). \tag{13}$$

The relation (13) determines the bandwidth of the hypothetical resonance which is narrower than that considered in previous works [11, 13, 15]. This is due to the fact that energy transfer to cross waves occurs through higher-order interactions. The bandwith (13) anticipates the scaling of slow variables and the form of the averaged Lagrangian.

The crucial point of the analysis is the choice of a trial solution for the total potential ϕ. Taking into consideration the results of our experimental measurements, we pose the trial functions in dimensionless form:

$$\frac{k^2}{\omega}\phi = \varepsilon(\phi_0 + \phi_1) + \varepsilon^2(\phi_{00} + \phi_{01} + \phi_{11}) + \varepsilon^3(\phi_{000} + \phi_{001} + \phi_{011} + \phi_{111}) + \cdots, \tag{14}$$

$$k\eta = \varepsilon(\eta_0 + \eta_1) + \varepsilon^2(\eta_{00} + \eta_{01} + \eta_{11}) + \varepsilon^3(\eta_{000} + \eta_{001} + \eta_{011} + \eta_{111}) + \cdots, \tag{15}$$

where: $(\phi_0,\ \eta_0)$ represents the linearized forced wave, $(\phi_1,\ \eta_1)$ describes the linear approximation to the cross wave, ϕ_{jp}, η_{jp} represent the interactions among first-order wave components, and ϕ_{jpq}, η_{jpq} are third-order wave components. The presence of the third-order terms in the expansions (14), (15) is due to the necessity of including of all fourth-order terms in the functional (11).

For the purpose of further analysis it is also necessary to expand the function (3) describing the instantaneous position of the wavemaker in a Taylor series about its rest position

$$k\xi = \varepsilon\xi_0 + \varepsilon^2\xi_{00} + O(\varepsilon^3). \tag{16}$$

The expansion components are

$$\xi_0 = \mathfrak{Re}\{i\ \mathrm{e}^{-2i\theta}\}\cos\vartheta, \qquad \xi_{00} = (\mathfrak{Re}\{\mathrm{e}^{-4i\theta}\} - 1)(1-\cos 2\vartheta)/[8r_1]. \tag{17}$$

Inserting (14), (15) and (16) into the Equations (4)–(8), we obtain the governing equations for the components of ϕ and η.

The first-order boundary-value problems are described by

$$\nabla^2\phi_j = 0, \quad \text{(in fluid)}, \tag{18}$$

$$\phi_{j,z} - \eta_{j,\theta} = 0, \qquad \phi_{j,\theta} + T^{-1}\eta_j = 0, \quad (z=0), \tag{19}$$

$$\phi_{j,r} = \delta_{0j}\xi_{0,\theta}, \quad (r = r_1), \tag{20}$$

$$\phi_{j,z} = 0 \quad (z = -h). \tag{21}$$

The second-order approximation is given by

$$\nabla^2\phi_{jp} = 0, \quad \text{(in fluid)}, \tag{22}$$

$$\begin{aligned}\phi_{jp,z} - \eta_{jp,\theta} &= \left(1 - \frac{\delta_{jp}}{2}\right)(\nabla\phi_j\nabla\eta_p + \nabla\phi_p\nabla\eta_j - \eta_j\phi_{p,zz} - \eta_p\phi_{j,zz}),\\ \phi_{jp,\theta} + T^{-1}\eta_{jp} &= \left(\frac{\delta_{jp}}{2} - 1\right)(\eta_j\phi_{p,\theta z} + \eta_p\phi_{j,\theta z} + \nabla\phi_j\nabla\phi_p), \quad (z=0),\end{aligned} \tag{23}$$

$$\phi_{jp,r} = \xi_{jp,\theta} + \nabla\xi_j\nabla\phi_p - \xi_j\phi_{p,rr}, \quad (r = r_1), \tag{24}$$

$$\phi_{jp,z} = 0 \quad (z = -h). \tag{25}$$

The third-order boundary-value problems are described by

$$\nabla^2\phi_{jpq} = 0, \quad \text{(in fluid)}, \tag{26}$$

$$\begin{aligned}\phi_{jpq,z} - \eta_{jpq,\theta} \ =\ & \left(1 - \frac{\delta_{1j}}{2}\right)(\nabla\phi_{jp}\nabla\eta_q + \nabla\phi_q\nabla\eta_{jp} + \nabla\phi_{pq}\nabla\eta_j\\ & + \nabla\phi_j\nabla\eta_{pq} - \eta_{jp}\phi_{q,zz} - \eta_q\phi_{jp,zz} - \eta_{pq}\phi_{j,zz} - \eta_j\phi_{pq,zz})\\ & -\tfrac{1}{2}\eta_p^2\phi_{(j-p+q),zzz} + (1 - \tfrac{2}{3}\delta_{1j})(\eta_j\nabla\phi_{p,z}\nabla\eta_q\\ & + \eta_p\nabla\phi_{q,z}\nabla\eta_j + \eta_q\nabla\phi_{j,z}\nabla\eta_p), \quad (z=0),\end{aligned} \tag{27}$$

$$
\begin{aligned}
\phi_{jpq,\theta} + T^{-1}\eta_{jpq} \;=\; & \left(\frac{\delta_{1j}}{2} - 1\right)(\nabla\phi_{jp}\nabla\phi_q + \nabla\phi_{pq}\nabla\phi_j \\
& + \eta_{jp}\phi_{q,\theta z} + \eta_q\phi_{jp,\theta z} + \eta_{pq}\phi_{j,\theta z} + \eta_j\phi_{pq,\theta z}) \\
& - \tfrac{1}{2}\eta_p^2\phi_{(j-p+q),\theta zz} + (\tfrac{2}{3}\delta_{1j} - 1)(\eta_j\nabla\phi_{p,z}\nabla\phi_q \\
& + \eta_p\nabla\phi_{q,z}\nabla\phi_j + \eta_q\nabla\phi_{j,z}\nabla\phi_p), \quad (z = 0),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\phi_{jpq,r} \;=\; & -\xi_{jp}\phi_{q,rr} + \nabla\xi_j\nabla\phi_{pq} - \xi_j\phi_{pq,rr} + \xi_j\nabla\xi_p\nabla\phi_q \\
& - \tfrac{1}{2}\xi_j\xi_p\phi_{q,rrr}, \quad (r = r_1),
\end{aligned}
\tag{29}
$$

$$
\phi_{jpq,z} = 0 \quad (z = -h). \tag{30}
$$

In the Equations (18)–(30) we have $j,\ p,\ q = 0, 1$; δ_{jp} is the Kronecker delta and T denotes $\tanh kh$. Moreover, $\xi_1 \equiv \xi_{01} \equiv \xi_{11} \equiv 0$. The Equations (18)–(30) have been derived on the assumption (13) from which the following approximation follows

$$
\frac{\omega_k^2}{\omega^2} = \frac{kg\tanh kh}{\omega^2} = 1 - O(\varepsilon^4) \approx 1. \tag{31}
$$

4. First-order problems

It has already been mentioned that the explicit solutions to all first-order problems considered here are required in the subsequent analysis.

The linear approximation to the forced wave ($j = 0$) can be found through a Hankel integral transform with respect to the radial coordinate. Proceeding similarly to Becker and Miles [15], we anticipate a solution of the form

$$
[\phi_0, \eta_0] = \Re\mathfrak{e}\left\{[\Phi_0(r,z),\ iZ_0(r)]\,\mathrm{e}^{-2i\theta}\right\}\cos\vartheta \tag{32}
$$

and define the following Hankel-transform pair

$$
\hat{\Phi}_0(z,\mu) = \int_{r_1}^{\infty} \Phi_0(r,z)F_1(\mu r, \mu r_1)r\,\mathrm{d}r, \tag{33}
$$

$$
\Phi_0(r,z) = \int_0^{\infty} \hat{\Phi}_0(z,\mu)F_1(\mu r, \mu r_1)\mu\,\mathrm{d}\mu \tag{34}
$$

for the complex amplitude $\Phi_0(r,z)$ and similarly for $Z_0(r)$. The function $F_1(\mu r, \mu r_1)$ satisfies the homogeneous boundary condition (20) on the wavemaker (cylinder) surface $r = r_1$ and can be expressed in terms of Bessel and Hankel functions of the first order as follows

$$
\begin{aligned}
F_1(\mu r, \mu r_1) \;=\; & \frac{J_1(\mu r)Y_1'(\mu r_1) - Y_1(\mu r)J_1'(\mu r_1)}{[Y_1'^2(\mu r_1) + J_1'^2(\mu r_1)]^{1/2}} \\
=\; & \frac{H_1^{(2)}(\mu r)H_1^{(1)'}(\mu r_1) - H_1^{(1)}(\mu r)H_1^{(2)'}(\mu r_1)}{2i[H_1^{(1)'}(\mu r_1)H_1^{(2)'}(\mu r_1)]^{1/2}},
\end{aligned}
\tag{35}
$$

where primes signify differentation with respect to the argument.

Inserting (32) into (18)–(21), carrying out the transformation according to (33), solving the resulting differential equation with the proper boundary conditions for the transforms, and carrying out the inverse transform (34), we obtain the first-order solution for the forced wave in finite depth h

$$\Phi_0 = \frac{2}{\pi i}\int_0^\infty \left[\frac{H_1^{(1)}(\mu r)}{H_1^{(1)'}(\mu r_1)} - \frac{H_1^{(2)}(\mu r)}{H_1^{(2)'}(\mu r_1)}\right]\left(1+\frac{4\cosh\mu(z+h)}{\cosh\mu h(\mu T-4)}\right)\frac{\mathrm{d}\mu}{\mu^2},$$

$$Z_0 = \frac{4}{\pi i}\int_0^\infty \left[\frac{H_1^{(1)}(\mu r)}{H_1^{(1)'}(\mu r_1)} - \frac{H_1^{(2)}(\mu r)}{H_1^{(2)'}(\mu r_1)}\right]\left(\frac{T}{\mu T-4}\right)\frac{\mathrm{d}\mu}{\mu}, \tag{36}$$

where $T = \tanh\mu h$ and the path of integration is deformed under the real pole μ_0 ($\mu_0 T - 4 = 0$) in order to satisfy the radiation condition at $r = \infty$. This solution comprises both radiated waves (contribution from $H_1^{(1)}(\mu r)$) and evanescent modes (contribution from $H_1^{(1)}(\mu r)$ and $H_1^{(2)}(\mu r)$).

Evaluating the integrals in (36), we obtain a well-known solution

$$\Phi_0 = \frac{4H_1^{(1)}(\mu_0 r)}{\mu_0 H_1^{(1)'}(\mu_0 r_1)}\frac{\sinh 2\mu_0 h}{(2\mu_0 h + \sinh 2\mu_0 h)}\frac{\cosh\mu_0(z+h)}{\cosh\mu_0 h}$$

$$+\sum_{\ell=1}^{\infty}\frac{4K_1(\kappa_\ell r)}{\kappa_\ell K_1'(\kappa_\ell r_1)}\frac{\sin 2\kappa_\ell h}{(2\kappa_\ell h + \sin 2\kappa_\ell h)}\frac{\cos\kappa_\ell(z+h)}{\cos\kappa_\ell h}, \tag{37}$$

where $i\kappa_\ell$ are the imaginary poles of the integrand in (36).

When water depth increases ($h \to \infty,\ T \to 1$), the radiated component of the solution (37) reduces to

$$\Phi_0^{(r)} = \frac{H_1^{(1)}(4r)}{H_1^{(1)'}(4r_1)}\,\mathrm{e}^{4z}, \tag{38}$$

and the local components (evanescent modes) are given by

$$\Phi_0^{(e)} = \lim_{\substack{h\to\infty \\ \kappa_\ell h \to \frac{(2\ell-1)\pi}{2}}}\sum_{\ell=1}^{\infty}\frac{8K_1(\kappa_\ell r)}{\kappa_\ell K_1'(\kappa_\ell r_1)}\frac{\sin\kappa_\ell h}{(2\kappa_\ell h + \sin 2\kappa_\ell h)}\cos\kappa_\ell(z+h)$$

$$= -\frac{8r_1^2}{r}\sum_{\ell=1}^{\infty}\frac{\sin[(2l-1)\pi/2]}{(2\ell-1)\pi}\cos\left(\frac{(2l-1)\pi}{2h}(z+h)\right), \tag{39}$$

which converges to zero on the free surface and to $-2r_1^2/r$ on the bottom.

Hence, evaluating the integrals over the free surface, we need only the radiated component (38) of the solution for Φ_0 that is $O(r^{-1/2})$. The calculation of integrals over the cylinder surface seems to require both radiated and evanescent ($O(r^{-1})$) components, even for a deep-water approximation which is relevant for the problem considered. However, further analysis will reveal that a part of the cylinder-surface integral can be transformed into a free-surface integral and the remaining part gives a null contribution. Therefore, the evanescent modes (39) do not contribute to the functional (11) and can be neglected in subsequent analysis.

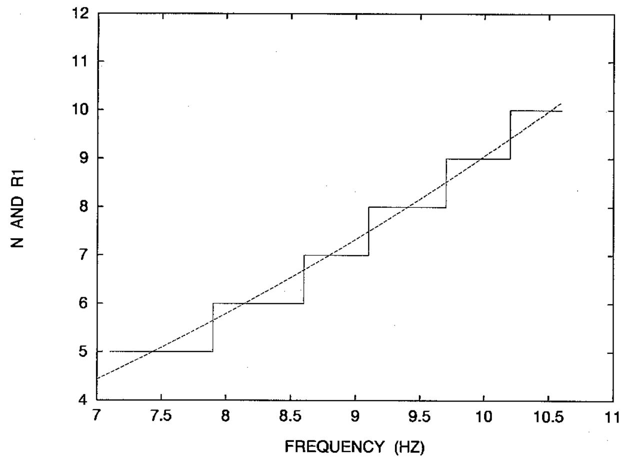

Figure 3. Comparison of n (stepped line) and r_1 (dotted line) for a circular cylinder.

The linear approximation to the cross wave ($j = 1$) has to satisfy Equations (18) and (19), together with the homogeneous boundary condition (20) on the cylinder. We use a deep-water approximation and require, instead of (21), that ϕ_1 vanishes if $z \to -\infty$.

Since the problem considered here is not axisymmetric, we choose the solution for the cross wave in the general form

$$[\phi_1,\ \eta_1] = \sqrt{2}F_n(r, r_1)\Big[\mathfrak{Re}\{[-i\ \mathrm{e}^z,\ 1]\mathbf{A}_c(R,\tau)\,\mathrm{e}^{-i\theta}\}\cos n\vartheta + \mathfrak{Re}\{[-i\ \mathrm{e}^z,\ 1]\mathbf{A}_s(R,\tau)\,\mathrm{e}^{-i\theta}\}\sin n\vartheta\Big], \tag{40}$$

where $\mathbf{A}_c(R,\tau)$ and $\mathbf{A}_s(R,\tau)$ are dimensionless, slowly varying complex amplitudes; n is an azimuthal wavenumber and $R = 2\varepsilon^2 r$ and $\tau = \varepsilon^4\theta$ are slow variables (see [15], where $R = 2\varepsilon r$ and $\tau = \varepsilon^2\theta$). The amplitudes $\mathbf{A}_c(R,\tau)$ and $\mathbf{A}_s(R,\tau)$ are to be calculated from the evolution equations.

The comprehensive discussion of the properties of radial cross waves is given in the paper cited above; here we remark only that, since the energy is transferred from the wavemaker to the cross wave through weak nonlinear interactions, the cross wave must be a standing wave in the first (linear) approximation. Therefore, $F_n(r, r_1)$ must have the form (35), where $\mu \equiv 1$ and where Bessel and Hankel functions of order 1 have to be replaced by the same functions of order n. Moreover, the cross wave excitation is most efficient at that wavenumber n for which the turning point of Bessel's equation is at the cylinder. Thus, we may expect n be of order $O(r_1)$ (r_1 – dimensionless cylinder radius). This assumption has already been confirmed in experiments of our own with different cylindrical wavemakers. For instance, the comparison of the observed values of n with r_1 for a circular cylinder (radius – 9 cm) is given in Figure 3. It should also be noted that, in view of the parameter range considered ($r_1 \in (4, 12)$), the Hankel functions in (38) and (40) can be replaced by their asymptotic approximations, even near the cylinder surface.

5. Second-order problems

The solution to all second-order problems described by the Equations (22)–(25) can also be found through Hankel transforms. The calculations are straightforward in principle (though tedious) and follow the same line for all involved second-order interactions ϕ_{jp}. Our analysis will show, however, that only (ϕ_{01}, η_{01}) and (ϕ_{11}, η_{11}) are explicitly required for the the derivation of the evolution equations. These two components admit the use of the deep-water approximation and we shall take advantage of this to simplify the results. Below we present the solution procedure and some results in a compact form.

Inserting the solutions of the first-order problems into the Equations (22)–(25), we obtain on their right-hand sides a combination of terms with various harmonic components in the time domain (index m) and in space (index α). There appear terms proportional to $\mathbf{A}_c$, $\mathbf{A}_s$, $\mathbf{A}_c^2$, $\mathbf{A}_s^2$ and to $\mathbf{A}_c\mathbf{A}_s$. There are also terms which do not depend on the amplitudes. Eventually, we obtain the following boundary-value problem for each harmonic component m $(m = 0, 1, 2, 3, 4)$ of the potential function ϕ_{jp}

$$\nabla^2 \phi_{jp}^m = \delta(r - r_1) \sum_{l \in \mathcal{S}} \mathfrak{Re} \left\{ \mathcal{A}_l^m Q^{ml}(r_1, \vartheta, z)\, \mathrm{e}^{-im\theta} \right\} \quad \text{(in fluid)},$$

$$\left. \begin{aligned} \phi_{jp,z}^m - \eta_{jp,\theta}^m &= \sum_{l \in \mathcal{S}} \mathfrak{Re} \left\{ \mathcal{A}_l^m G^{ml}(r, \vartheta)\, \mathrm{e}^{-im\theta} \right\}, \\ \phi_{jp,\theta}^m + \eta_{jp}^m &= \sum_{l \in \mathcal{S}} \mathfrak{Re} \left\{ \mathcal{A}_l^m H^{ml}(r, \vartheta)\, \mathrm{e}^{-im\theta} \right\}, \end{aligned} \right\} \quad (z = 0), \tag{41}$$

$$\phi_{jp,r}^m = 0 \quad (r = r_1), \qquad \phi_{jp,z}^m \to 0 \quad (z \to -\infty), \tag{42}$$

where $\delta(r - r_1)$ is the Kronecker delta, and the following sets $\mathcal{S}$ and $\mathcal{A}_l^m$ have been defined for the second-order problems considered:

$$\begin{aligned} &\mathcal{S} = \{s,\, c\}, \qquad \mathcal{A}_l^1 = \mathbf{A}_l^*, \qquad \mathcal{A}_l^3 = \mathbf{A}_l \quad \text{for the case } (01), \\ &\mathcal{S} = \{s,\, c,\, sc\}, \qquad \mathcal{A}_s^0 = |\mathbf{A}_s|^2, \qquad \mathcal{A}_c^0 = |\mathbf{A}_c|^2, \qquad \mathcal{A}_{sc}^0 = \mathbf{A}_s\mathbf{A}_c^* + \mathbf{A}_c\mathbf{A}_s^*, \\ &\mathcal{A}_s^2 = \mathbf{A}_s^2, \qquad \mathcal{A}_c^2 = \mathbf{A}_c^2, \qquad \mathcal{A}_{sc}^2 = 2\mathbf{A}_s\mathbf{A}_c \quad \text{for the case } (11). \end{aligned} \tag{43}$$

Omitting here the details of the functions Q^{ml}, G^{ml} and H^{ml}, we pose the solution for each problem (jp) in the form

$$[\phi_{jp}^m, \eta_{jp}^m] = (2)^{p/2} \sum_{l \in \mathcal{S}} \mathfrak{Re} \left\{ \mathcal{A}_l^m [i\Phi^{ml}(r, \vartheta, z),\, Z^{ml}(r, \vartheta)]\, \mathrm{e}^{-mi\theta} \right\} \tag{44}$$

and define the following Hankel-transform pair

$$\begin{bmatrix} \hat{\Phi}^{ml} \\ \hat{Z}^{ml} \end{bmatrix}_\alpha = \int_0^{2\pi} \int_{r_1}^{\infty} \begin{bmatrix} \Phi^{ml} \\ Z^{ml} \end{bmatrix} \mathcal{F}_\alpha^l(\mu r, \mu r_1) r \,\mathrm{d}r\, \mathrm{d}\vartheta, \tag{45}$$

$$\begin{bmatrix} \Phi^{ml} \\ Z^{ml} \end{bmatrix} = \sum_\alpha \frac{2 - \delta_{0\alpha}}{2\pi} \int_0^{\infty} \begin{bmatrix} \hat{\Phi}^{ml} \\ \hat{Z}^{ml} \end{bmatrix}_\alpha \mathcal{F}_\alpha^l(\mu r, \mu r_1)\mu \,\mathrm{d}\mu, \tag{46}$$

where the following identities hold for the different second-order waves:

– the case (01)

$$\mathcal{F}_\alpha^s \equiv F_\alpha(\mu r, \mu r_1) \sin \alpha\vartheta, \qquad \mathcal{F}_\alpha^c \equiv F_\alpha(\mu r, \mu r_1) \cos \alpha\vartheta$$

$$\text{for} \quad \alpha = (n-1), (n+1),$$

– the case (11)

$$\mathcal{F}_\alpha^s \equiv F_\alpha(\mu r, \mu r_1)\cos(\delta_{(2n)\alpha}\pi - \alpha\vartheta), \qquad \mathcal{F}_\alpha^c \equiv F_\alpha(\mu r, \mu r_1)\cos\alpha\vartheta,$$
$$\mathcal{F}_\alpha^{sc} \equiv F_\alpha(\mu r, \mu r_1)\sin\alpha\vartheta \quad \text{for} \quad \alpha = 0,\ 2n,$$

and $F_\alpha(\mu r, \mu r_1)$ is given by Equation (35) modified for Bessel functions of order α.

Inserting (44) into (41) and carrying out the integral transformation, we obtain the boundary-value problems for the transforms of each space-harmonic component α:

$$\hat{\Phi}^{ml}_{\alpha,zz} - \mu^2\hat{\Phi}^{ml}_\alpha = Q^{ml}_\alpha(r_1, z)F_\alpha(\mu r_1, \mu r_1)r_1 \quad \text{(in fluid)}, \tag{47}$$

$$\left.\begin{aligned} \hat{\Phi}^{ml}_{\alpha,z} + m\,\hat{Z}^{ml}_\alpha &= \hat{\mathcal{G}}^{ml}_\alpha \\ m\,\hat{\Phi}^{ml}_\alpha + \hat{Z}^{ml}_\alpha &= \hat{\mathcal{H}}^{ml}_\alpha \end{aligned}\right\} \quad (z = 0), \tag{48}$$

$$\hat{\Phi}^{ml}_{\alpha,r} = 0 \quad (r = r_1), \qquad \hat{\Phi}^{ml}_{\alpha,z} \to 0 \quad (z =\to -\infty). \tag{49}$$

The solution to this system of equations is

$$\begin{aligned} \hat{\Phi}^{ml}_\alpha &= \frac{\hat{\mathcal{G}}^{ml}_\alpha - m\,\hat{\mathcal{H}}^{ml}_\alpha}{\mu - m^2}\,\mathrm{e}^{\mu z} - \frac{1}{2\mu}\int_{-\infty}^{0} f^{ml}_\alpha(\nu)\left(\mathrm{e}^{-\mu|z-\nu|} + \frac{\mu + m^2}{\mu - m^2}\,\mathrm{e}^{\mu(z+\nu)}\right)\,\mathrm{d}\nu, \\ \hat{Z}^{ml}_\alpha &= \frac{m}{\mu - m^2}\left(\frac{\mu}{m}\hat{\mathcal{H}}^{ml}_\alpha - \hat{\mathcal{G}}^{ml}_\alpha + \int_{-\infty}^{0} f^{ml}_\alpha(\nu)\,\mathrm{e}^{\mu\nu}\,\mathrm{d}\nu\right), \end{aligned} \tag{50}$$

where $f^{ml}_\alpha(\nu) = Q^{ml}_\alpha(r_1, \nu)F_\alpha(\mu r_1, \mu r_1)r_1$.

The solution for each second-order wave (ϕ_{jp}, η_{jp}) can be obtained with the use of the inverse integral transform (46). For instance, for a second-order cross wave we obtain:

$$\begin{aligned} \begin{bmatrix} \phi_{11} \\ \eta_{11} - \langle\eta_{11}\rangle \end{bmatrix} &= \mathfrak{Re}\left\{\tfrac{1}{2}\,\mathrm{e}^{-2i\theta}\sum_{\alpha=0,2n}\int_0^\infty \begin{bmatrix} i(\mu^2 - 4)\,\mathrm{e}^{\mu z} \\ 2\mu - \mu^2 - \tfrac{1}{4}\mu^3 \end{bmatrix} \mathcal{K}^{(11)}_\alpha(\mu)F_\alpha(\mu r, \mu r_1)\frac{\mu\,\mathrm{d}\mu}{\mu - 4}\right. \\ &\quad \left.\cdot\left[\mathbf{A}_s^2\cos(\delta_{(2n)\alpha}\pi - \alpha\vartheta) + \mathbf{A}_c^2\cos\alpha\vartheta + 2\mathbf{A}_s\mathbf{A}_c\sin\alpha\vartheta\right]\right\}, \end{aligned} \tag{51}$$

where $\mathcal{K}^{(11)}_\alpha(\mu) = \int_{r_1}^\infty F_n^2(r, r_1)F_\alpha(\mu r, \mu r_1)r\,\mathrm{d}r$, and the temporal mean wave elevation is

$$\begin{aligned} \langle\eta_{11}\rangle &= \tfrac{1}{2}\{|\mathbf{A}_s|^2[(\mathcal{F}_n^s)^2 - (\nabla\mathcal{F}_n^s)^2] + |\mathbf{A}_c|^2[(\mathcal{F}_n^c)^2 - (\nabla\mathcal{F}_n^c)^2] \\ &\quad + (\mathbf{A}_s\mathbf{A}_c^* + \mathbf{A}_c\mathbf{A}_s^*)[\mathcal{F}_n^s\mathcal{F}_n^c - \nabla\mathcal{F}_n^s\nabla\mathcal{F}_n^c]\}. \end{aligned} \tag{52}$$

The path of integration in (51) passes under the pole at $\mu = 4$ in order to satisfy the radiation condition at $r = \infty$. The asymptotic approximation for $r \to \infty$ is dominated by the contribution of this pole and its complex amplitude is given by

$$\begin{aligned} \begin{bmatrix} \Phi_{11} \\ Z_{11} \end{bmatrix} &\sim \begin{bmatrix} 12i\,\mathrm{e}^{4z} \\ -24 \end{bmatrix}\left(\frac{2\pi}{r}\right)^{1/2} \mathrm{e}^{i(4r + \frac{3}{4}\pi)}\sum_{\alpha=0,2n}\mathcal{D}_\alpha(4r_1)\mathcal{K}^{(11)}_\alpha(4) \\ &\quad \cdot\left[\mathbf{A}_s^2\cos(\delta_{(2n)\alpha}\pi - \alpha\vartheta) + \mathbf{A}_c^2\cos\alpha\vartheta + 2\mathbf{A}_s\mathbf{A}_c\sin\alpha\vartheta\right] + O(r^{-1}), \end{aligned} \tag{53}$$

where

$$\mathcal{D}_\alpha(4r_1) = \left(\frac{H_\alpha^{(2)'}(4r_1)}{H_\alpha^{(1)'}(4r_1)}\right)^{1/2} \mathrm{e}^{i\pi\alpha/2}.$$

The results for ϕ_{01} and η_{01} are:

$$\phi_{01} = \sqrt{2}\,\mathfrak{Re}\left\{(\mathbf{A}_s^*\Phi^{1s} + \mathbf{A}_c^*\Phi^{1c})i\,\mathrm{e}^{-i\theta} + (\mathbf{A}_s\Phi^{3s} + \mathbf{A}_c\Phi^{3c})i\,\mathrm{e}^{-3i\theta}\right\}, \tag{54}$$

$$\eta_{01} = \sqrt{2}\,\mathfrak{Re}\left\{(\mathbf{A}_s^*Z^{1s} + \mathbf{A}_c^*Z^{1c})\,\mathrm{e}^{-i\theta} + (\mathbf{A}_sZ^{3s} + \mathbf{A}_cZ^{3c})\,\mathrm{e}^{-3i\theta}\right\}. \tag{55}$$

The complex amplitudes $\Phi^{1\ell}$, $Z^{1\ell}$, $\Phi^{3\ell}$ and $Z^{3\ell}$ are given by

$$\begin{aligned}\begin{bmatrix}\Phi^{1\ell}\\ Z^{1\ell}\end{bmatrix} = \sum_{\alpha=(n-1),(n+1)} \frac{i}{8}\int_0^\infty \Bigg\{&\begin{bmatrix}(50-2\mu^2)\,\mathrm{e}^{\mu z}\\ \mu^3+\mu^2-5\mu-45\end{bmatrix}\mathcal{K}_\alpha^{(01)}(\mu)\\ &+\begin{bmatrix}\mathrm{e}^z\\ -1\end{bmatrix}\mathcal{M}_\alpha(\mu)\Bigg\}\mathcal{F}_\alpha^\ell(\mu r, \mu r_1)\frac{\mu\,\mathrm{d}\mu}{\mu-1},\end{aligned} \tag{56}$$

$$\begin{aligned}\begin{bmatrix}\Phi^{3\ell}\\ Z^{3\ell}\end{bmatrix} = \sum_{\alpha=(n-1),(n+1)} -\frac{i}{8}\int_0^\infty \Bigg\{&\begin{bmatrix}3(50-2\mu^2)\,\mathrm{e}^{\mu z}\\ \mu^3+9\mu^2-45\mu-45\end{bmatrix}\mathcal{K}_\alpha^{(01)}(\mu)(\mu-1)\\ &+\begin{bmatrix}(\mu-9)\,\mathrm{e}^z+8\,\mathrm{e}^{\mu z}\\ -3(\mu-1)\end{bmatrix}\mathcal{M}_\alpha(\mu)\Bigg\}\mathcal{F}_\alpha^\ell(\mu r, \mu r_1)\frac{\mu\,\mathrm{d}\mu}{(\mu-1)(\mu-9)},\end{aligned} \tag{57}$$

where

$$\mathcal{M}_{(n\mp1)} = \frac{2r_1}{\mu+1}\left(F_{n,rr} \mp \frac{n}{r^2}F_n\right)_{|r=r_1} F_{(n\mp1)}(\mu r_1, \mu r_1),$$

$$\mathcal{K}_\alpha^{(01)} = \int_{r_1}^\infty \frac{H_1^{(1)}(4r)}{H_1^{(1)'}(4r_1)}F_n(r, r_1)F_\alpha(\mu r, \mu r_1)r\,\mathrm{d}r.$$

An asymptotic approximation to (56) and (57) satisfying the radiation condition at $r = \infty$ is

$$\begin{aligned}\begin{bmatrix}\Phi^{m\ell}\\ Z^{m\ell}\end{bmatrix} &\sim \frac{i}{8}(-1)^{\frac{m+1}{2}}\begin{bmatrix}\mathrm{e}^{m^2z}\\ -m\end{bmatrix}\left(\frac{2\pi m^2}{r}\right)^{1/2}\mathrm{e}^{i(m^2r+3\pi/4)}\\ &\sum_{\alpha=(n-1),(n+1)}\mathcal{D}_\alpha^\ell(m^2r_1)\left[(2m^4-50)m\mathcal{K}_\alpha^{(01)}(m^2) - \mathcal{M}_\alpha(m^2)\right] + \begin{bmatrix}O(r^{-1})\\ O(r^{-3/2})\end{bmatrix},\end{aligned} \tag{58}$$

where

$$\mathcal{D}_\alpha^s(\cdot) = \left(\frac{H_\alpha^{(2)'}(\cdot)}{H_\alpha^{(1)'}(\cdot)}\right)^{1/2}\mathrm{e}^{-i\pi\alpha/2}\sin\alpha\vartheta, \qquad \mathcal{D}_\alpha^c(\cdot) = \left(\frac{H_\alpha^{(2)'}(\cdot)}{H_\alpha^{(1)'}(\cdot)}\right)^{1/2}\mathrm{e}^{-i\pi\alpha/2}\cos\alpha\vartheta.$$

6. Average Lagrangian

The solutions obtained for the first- and second-order waves depend both on the fast and on the slow variables:

$$\phi_0 = \phi_0(r, \vartheta, z), \qquad \phi_1 = \phi_1(r, \vartheta, z, R, \tau),$$

$$\phi_{00} = \phi_{00}(r, \vartheta, z), \qquad \phi_{01} = \phi_{01}(r, \vartheta, z, R, \tau), \qquad \phi_{11} = \phi_{11}(r, \vartheta, z, R, \tau).$$

Similar relations are valid for η_j and η_{jp}.

In order to calculate the averaged Lagrangian, we insert the trial solutions (14) and (15) into the functional (11), including all terms up to the order $O(\varepsilon^4)$. Proceeding similarly to Becker and Miles [15], we expand the integrands in the free-surface and wavemaker integrals about $z = 0$ and $r = r_1$, respectively, separate out the contribution of the end point in the free-surface integral by the approximation

$$\int_{r_0}^{\infty} [\sim]\, \mathrm{d}r \approx \int_{r_1}^{\infty} [\sim]\, \mathrm{d}r - r_0 [\sim]_{r=r_1},$$

where $r_0 \approx \xi(\vartheta, 0, t)$, apply the same approximation to the wavemaker integral, and carry out the differentiation (where necessary) according to the rule

$$\frac{\partial}{\partial r} \mapsto \frac{\partial}{\partial r} + 2\varepsilon^2 \frac{\partial}{\partial R}, \qquad \frac{\partial^2}{\partial r^2} \mapsto \frac{\partial^2}{\partial r^2} + 4\varepsilon^2 \frac{\partial^2}{\partial r \partial R} + 4\varepsilon^4 \frac{\partial^2}{\partial R^2},$$

$$\frac{\partial}{\partial \theta} \mapsto \frac{\partial}{\partial \theta} + \varepsilon^4 \frac{\partial}{\partial \tau}.$$

Further, we separate out the Lagrangian L_0 of the forced wave which is indepenent of $\mathbf{A}_s$ and $\mathbf{A}_c$ and does not contribute to Hamilton's principle. Finally, we average $\langle\ \rangle$ the Lagrangian over the fast time θ to obtain the dimensionless average Lagrangian in the form

$$\mathcal{L} = \frac{2k^5}{\omega^2 \varepsilon^4} \langle L - L_0 \rangle = \mathcal{L}_{11} + \mathcal{L}_{0011} + \mathcal{L}_{1111} + O(\varepsilon^2). \tag{59}$$

The components of the averaged Lagrangian are:

$$\begin{aligned}
\mathcal{L}_{11} \;=\; & 2\int_0^{2\pi} \int_{-h}^{0} \int_{r_1}^{\infty} \left\langle \phi_1(\phi_{1,R} + 2r\phi_{1,rR} + 2\varepsilon^2 r \phi_{1,RR}) \right\rangle \,\mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\vartheta \\
& +2\int_0^{2\pi} \int_{-h}^{0} \left\langle \phi_1 \phi_{1,R} r \right\rangle_{r=r_1} \,\mathrm{d}z\, \mathrm{d}\vartheta \\
& + \int_0^{2\pi} \int_{R_1}^{\infty} \left\langle \phi_1 \eta_{1,\tau} + \frac{1}{2\varepsilon^4} \left(\phi_1 \eta_{1,\theta} - \frac{kg}{\omega^2} \eta_1^2 \right) \right\rangle_{z=0} r\, \mathrm{d}R\, \mathrm{d}\vartheta,
\end{aligned} \tag{60}$$

$$\begin{aligned}
\mathcal{L}_{0011} \;=\; & \int_0^{2\pi} \int_{r_1}^{\infty} \langle -\eta_1(\phi_{001,\theta} + \eta_{001}) - \eta_0(\phi_{011,\theta} + \eta_{011}) + \tfrac{1}{2}\eta_{11}(\nabla\phi_0)^2 \\
& +\eta_{01}\nabla\phi_0\nabla\phi_1 + \tfrac{1}{2}\eta_{00}(\nabla\phi_1)^2 + \phi_{11,z}\eta_0\eta_{0,\theta} + \phi_{01,z}(\eta_0\eta_1)_{,\theta} \\
& +\phi_{00,z}\eta_1\eta_{1,\theta} + \tfrac{1}{2}(\phi_{0,zz}\eta_1^2\eta_{0,\theta} + \phi_{1,zz}\eta_0^2\eta_{1,\theta}) \rangle_{z=0}\, r\, \mathrm{d}r\, \mathrm{d}\vartheta
\end{aligned}$$

$$- \int_0^{2\pi} \int_{-h}^{0} \langle (\phi_{011}\xi_{0,\theta} + \phi_{11}\xi_{00,\theta}) r_1 + \phi_{11}\xi_0\xi_{0,\theta} \rangle_{r=r_1} \, \mathrm{d}z \, \mathrm{d}\vartheta$$

$$\begin{aligned} - \int_0^{2\pi} \Big\langle & \eta_1\phi_1\xi_0\xi_{0,\theta} + \frac{r_1}{2}\xi_{0,\theta}\eta_1^2\phi_{0,z} + 2r_1\xi_0^2\phi_{1,r}\eta_{1,\theta} \\ & + r_1\xi_0 \big[(\phi_{11,\theta} + \eta_{11})(\phi_0 - \eta_0) - \eta_1(\phi_{01,\theta} + \eta_{01}) + \phi_{0,z}\eta_1\eta_{1,\theta} \\ & + \phi_{1,z}(\eta_0\eta_1)_{,\theta} \big] \Big\rangle_{\substack{r=r_1 \\ z=0}} \, \mathrm{d}\vartheta, \end{aligned} \tag{61}$$

$$\begin{aligned} \mathcal{L}_{1111} = \int_0^{2\pi} \int_{r_1}^{\infty} \langle & -\eta_1(\phi_{111,\theta} + \eta_{111}) - \eta_{11}(\phi_{11,\theta} + \eta_{11}) + \phi_{1,z}(\eta_1\eta_{11})_{,\theta} \\ & + \phi_{11,z}\eta_1\eta_{1,\theta} + \tfrac{1}{2}\phi_{1,zz}\eta_1^2\eta_{1,\theta} \rangle_{z=0} \, r \, \mathrm{d}r \, \mathrm{d}\vartheta. \end{aligned} \tag{62}$$

It should be noticed that in (61) and (62) the terms in parentheses are just the left-hand sides of the boundary conditions (23) and (28) for a deep-water approximation. Moreover, the integral of $\langle \phi_{011}\xi_{0,\theta} \rangle$ (the term of (61)) can be transformed with the use of Green's theorem and the boundary condition (29) according to

$$- \int_0^{2\pi} \int_{-h}^{0} \big\langle \phi_{011}\xi_{0,\theta} \big\rangle_{r=r_1} r \, \mathrm{d}z \, \mathrm{d}\vartheta = \int_0^{2\pi} \int_{r_1}^{\infty} \big\langle \phi_0\phi_{011,z} - \phi_{011}\phi_{0,z} \big\rangle_{z=0} r \, \mathrm{d}r \, \mathrm{d}\vartheta, \tag{63}$$

The relation (63) can further be combined with the term $-\eta_0(\phi_{011,\theta} + \eta_{011})$ of the functional (61) leading to

$$\big\langle \phi_0\phi_{011,z} - \phi_{011}\phi_{0,z} \big\rangle + \big\langle -\eta_0(\phi_{011,\theta} + \eta_{011}) \big\rangle = \big\langle \phi_0(\phi_{011,z} - \eta_{011,\theta}) \big\rangle,$$

where again the term in parentheses is identical with the left-hand side of the free-surface boundary condition. Therefore, all third-order wave components in (61) and (62) can be replaced by lower-order components, and the explicit solution to the third-order problems is not needed for the derivation of the evolution equations.

The functional (60) can be reduced to the form obtained by Becker and Miles [15]. Using the identities

$$\phi_1\phi_{1,rR} = \phi_{1,r}\phi_{1,R}, \qquad \langle \eta_1^2 \rangle = -\langle \eta_1\phi_{1,\theta} \rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\phi_1\phi_{1,R}) = \frac{\partial}{\partial r}(r\phi_1\phi_{1,R}) + 2\varepsilon^2\frac{\partial}{\partial R}(r\phi_1\phi_{1,R}),$$

and integrating by parts, we obtain

$$\mathcal{L}_{11} = \left\langle -2\int_0^{2\pi} \int_{-h}^{0} \int_{R_1}^{\infty} \phi_{1,R}^2 r \, \mathrm{d}R \, \mathrm{d}z \, \mathrm{d}\vartheta + \int_0^{2\pi} \int_{R_1}^{\infty} (\phi_1\eta_{1,\tau} + \beta\phi_1\eta_{1,\theta})_{z=0} r \, \mathrm{d}R \, \mathrm{d}\vartheta \right\rangle, \tag{64}$$

where $R_1 = 2\varepsilon^2 r_1$, and $\beta = (\omega^2 - \omega_k^2)/(2\varepsilon^4\omega^2)$. Clearly, β is of order $O(1)$ due to the resonance condition (13).

Inserting the solution for the first-order cross wave into (64), and averaging over the fast time θ, we have

$$\mathcal{L}_{11} = \Re\mathfrak{e}\left\{\int_{R_1}^{\infty}\{i(\mathbf{A}_c^*\mathbf{A}_{c,\tau} + \mathbf{A}_s^*\mathbf{A}_{s,\tau}) + \beta(\mathbf{A}_c\mathbf{A}_c^* + \mathbf{A}_s\mathbf{A}_s^*) - (\mathbf{A}_{c,R}\mathbf{A}_{c,R}^* + \mathbf{A}_{s,R}\mathbf{A}_{s,R}^*)\}\pi F_n^2(r)r\,\mathrm{d}R\right\}, \tag{65}$$

where the asterisk denotes a complex conjugate. Proceeding similarly to Becker and Miles [15], we use the asymptotic approximation to $\pi F_n^2(r)r$,

$$\pi F_n^2(r)r \sim 1 + \cos\left[\frac{R}{\varepsilon^2} - (n+\tfrac{1}{2})\pi + 2\tan^{-1}\left(\frac{J_n'(r_1)}{Y_n'(r_1)}\right)\right], \tag{66}$$

neglect the integrals with fast oscillating integrands, and approximate (65) by

$$\mathcal{L}_{11} = \Re\mathfrak{e}\left\{\int_{R_1}^{\infty}\{i(\mathbf{A}_c^*\mathbf{A}_{c,\tau} + \mathbf{A}_s^*\mathbf{A}_{s,\tau}) + \beta(\mathbf{A}_c\mathbf{A}_c^* + \mathbf{A}_s\mathbf{A}_s^*) - (\mathbf{A}_{c,R}\mathbf{A}_{c,R}^* + \mathbf{A}_{s,R}\mathbf{A}_{s,R}^*)\}\,\mathrm{d}R\right\}. \tag{67}$$

A similar procedure can be applied to the functionals (61) and (62). The oscillatory components of the integrands for $R = O(1)$ are neglected and the approximation $\mathbf{A} = \mathbf{A}(R_1, \tau) = \mathbf{A}_1$ for $r = O(r_1)$ is invoked. The result for $\mathcal{L}_{0011}$ is

$$\mathcal{L}_{0011} = \int_{R_1}^{\infty}\{\gamma_c\mathbf{A}_c\mathbf{A}_c^* + \gamma_s\mathbf{A}_s\mathbf{A}_s^*\}R^{-1}\,\mathrm{d}R + P_c\mathbf{A}_{1c}\mathbf{A}_{1c}^* + P_s\mathbf{A}_{1s}\mathbf{A}_{1s}^*. \tag{68}$$

The coefficients γ_c, γ_s for the deep-water approximation are

$$\gamma_c = \frac{175}{8\pi|H_1^{(1)'}(4r_1)|^2} \approx \frac{175}{4}r_1, \qquad \gamma_s = \frac{81}{8\pi|H_1^{(1)'}(4r_1)|^2} \approx \frac{81}{4}r_1. \tag{69}$$

The expressions for P_c and P_s are given in Appendix A.

The result of the approximation procedure for $\mathcal{L}_{1111}$ is

$$\mathcal{L}_{1111} = -\tfrac{1}{2}\delta\int_{R_1}^{\infty}(3a+b)R^{-1}\,\mathrm{d}R + \tfrac{1}{2}Q_1a_1 + \tfrac{1}{2}Q_2b_1 - Q_3\mathbf{A}_{1c}\mathbf{A}_{1c}^*\mathbf{A}_{1s}\mathbf{A}_{1s}^*, \tag{70}$$

where

$$a = (\mathbf{A}_c\mathbf{A}_c^* + \mathbf{A}_s\mathbf{A}_s^*)^2, \qquad b = (\mathbf{A}_c\mathbf{A}_s^* - \mathbf{A}_s\mathbf{A}_c^*)^2,$$

$$a_1 = a(R_1), \qquad b_1 = b(R_1), \qquad \delta = \frac{3}{8\pi}.$$

The expressions for Q_1, Q_2 and Q_3 are given in Appendix B. The detailed analysis shows (see Appendix B) that Q_3 is a quantity of higher order in comparison to Q_1 and Q_2 and can be neglected ($Q_3 = 0$).

Combining (65), (68) and (70), we arrive at the final form of the averaged Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; &\int_{R_1}^{\infty} \Big\{ \frac{i}{2}(\mathbf{A}_c^*\mathbf{A}_{c,\tau} - \mathbf{A}_c\mathbf{A}_{c,\tau}^*) + \frac{i}{2}(\mathbf{A}_s^*\mathbf{A}_{s,\tau} - \mathbf{A}_s\mathbf{A}_{s,\tau}^*) + \beta(\mathbf{A}_c\mathbf{A}_c^* + \mathbf{A}_s\mathbf{A}_s^*)\\ &\qquad + (\gamma_c\mathbf{A}_c\mathbf{A}_c^* + \gamma_s\mathbf{A}_s\mathbf{A}_s^*)R^{-1} - (\mathbf{A}_{c,R}\mathbf{A}_{c,R}^* + \mathbf{A}_{s,R}\mathbf{A}_{s,R}^*)\\ &\qquad - \tfrac{1}{2}\delta R^{-1}[3(\mathbf{A}_c\mathbf{A}_c^* + \mathbf{A}_s\mathbf{A}_s^*)^2 + (\mathbf{A}_c\mathbf{A}_s^* - \mathbf{A}_s\mathbf{A}_c^*)^2] \Big\}\, \mathrm{d}R\\ &+ P_c\mathbf{A}_{1c}\mathbf{A}_{1c}^* + P_s\mathbf{A}_{1s}\mathbf{A}_{1s}^* + \frac{Q_1}{2}(\mathbf{A}_{1c}\mathbf{A}_{1c}^* + \mathbf{A}_{1s}\mathbf{A}_{1s}^*)^2\\ &+ \frac{Q_2}{2}(\mathbf{A}_{1c}\mathbf{A}_{1s}^* - \mathbf{A}_{1s}\mathbf{A}_{1c}^*)^2 + O(\varepsilon^2). \end{aligned} \tag{71}$$

7. Evolution equations

The evolution equations for the parametrically excited cross wave can be derived from Hamilton's principle (9). Inserting (71) into (9), requiring the resulting functional to be stationary with respect to independent variations of $\mathbf{A}_c^*$ and $\mathbf{A}_s^*$, and invoking the null conditions at infinity ($\mathbf{A}_c = 0$, $\mathbf{A}_s = 0$ for $R \to \infty$), we obtain a set of two complex evolution equations

$$\mathbf{A}_{s,RR} + i\mathbf{A}_{s,\tau} + (\beta + \gamma_s R^{-1})\mathbf{A}_s - \delta R^{-1}\left(2|\mathbf{A}_c|^2\mathbf{A}_s + (3\mathbf{A}_s^2 + \mathbf{A}_c^2)\mathbf{A}_s^*\right) = 0, \tag{72}$$

$$\mathbf{A}_{c,RR} + i\mathbf{A}_{c,\tau} + (\beta + \gamma_c R^{-1})\mathbf{A}_c - \delta R^{-1}\left(2|\mathbf{A}_s|^2\mathbf{A}_c + (3\mathbf{A}_c^2 + \mathbf{A}_s^2)\mathbf{A}_c^*\right) = 0, \tag{73}$$

together with the boundary conditions at $R = R_1$

$$\mathbf{A}_{s,R} + P_s\mathbf{A}_s + Q_1\left(|\mathbf{A}_c|^2 + |\mathbf{A}_s|^2\right)\mathbf{A}_s + Q_2(\mathbf{A}_c\mathbf{A}_s^* - \mathbf{A}_s\mathbf{A}_c^*)\mathbf{A}_c = 0, \tag{74}$$

$$\mathbf{A}_{c,R} + P_c\mathbf{A}_c + Q_1\left(|\mathbf{A}_c|^2 + |\mathbf{A}_s|^2\right)\mathbf{A}_c - Q_2(\mathbf{A}_c\mathbf{A}_s^* - \mathbf{A}_s\mathbf{A}_c^*)\mathbf{A}_s = 0. \tag{75}$$

These equations have been derived on the assumption of a perfect fluid. Viscous dissipation may be incorporated through the transformation (see [12])

$$\mathbf{A}_{,\tau} \mapsto \mathbf{A}_{,\tau} + \alpha\mathbf{A}, \qquad \alpha = \zeta/\varepsilon^4,$$

where ζ is the damping ratio (on the scale θ) for the cross wave.

With this transformation, the evolution equations change into

$$\mathbf{A}_{s,RR} + i\mathbf{A}_{s,\tau} + \left(\beta + i\alpha + \frac{\gamma_s}{R}\right)\mathbf{A}_s - \frac{\delta}{R}\left(2|\mathbf{A}_c|^2\mathbf{A}_s + (3\mathbf{A}_s^2 + \mathbf{A}_c^2)\mathbf{A}_s^*\right) = 0, \tag{76}$$

$$\mathbf{A}_{c,RR} + i\mathbf{A}_{c,\tau} + \left(\beta + i\alpha + \frac{\gamma_c}{R}\right)\mathbf{A}_c - \frac{\delta}{R}\left(2|\mathbf{A}_s|^2\mathbf{A}_c + (3\mathbf{A}_c^2 + \mathbf{A}_s^2)\mathbf{A}_c^*\right) = 0, \tag{77}$$

without any change in the boundary conditions.

It is apparent that both the equations and the boundary conditions are coupled only through nonlinear terms. It is also obvious that they are satisfied by $\mathbf{A}_s = \mathbf{A}_c = 0$ which corresponds to a wave directly forced by the wavemaker. In order to determine the stability of this wave it is necessary to analyse the linearized boundary-value problem

$$\mathbf{A}_{s,RR} + i\mathbf{A}_{s,\tau} + (\beta + i\alpha + \gamma_s R^{-1})\mathbf{A}_s = 0, \tag{78}$$

$$\mathbf{A}_{c,RR} + i\mathbf{A}_{c,\tau} + (\beta + i\alpha + \gamma_c R^{-1})\mathbf{A}_c = 0 \tag{79}$$

with the boundary conditions

$$\mathbf{A}_{s,R} + P_s\mathbf{A}_s = 0, \quad (R = R_1), \qquad \mathbf{A}_s \to 0, \quad (R \to \infty), \tag{80}$$

$$\mathbf{A}_{c,R} + P_c\mathbf{A}_c = 0, \quad (R = R_1), \qquad \mathbf{A}_c \to 0, \quad (R \to \infty). \tag{81}$$

This boundary-value problem has some important features from which the qualitative behaviour of its possible solutions may be deduced. Firstly, the linear equations are uncoupled and can be analyzed separately. Secondly, in contrast to previous results (see [15]), there are two parameters γ and P (both are real numbers) describing the energy transfer from the wavemaker and the forced wave to the cross wave. The parameter γ plays an important role, since a simple analysis reveals that the trivial solution $\mathbf{A}_s = \mathbf{A}_c = 0$ is unconditionally stable when $\gamma = 0$. One should also notice that the flutter-type instability (not only divergence as in [15]) of the trivial solution is generally expected.

Finally, we may anticipate that, for a prescribed excitation frequency, the form of the cross wave emerging after the stability margin has been reached may be either symmetric or antisymmetric (with respect to the vertical plane of excitation), depending on the values of the parameters γ_c, γ_s, P_c and P_s. The results presented in Appendix A suggest, that due to the relationship $P_s > P_c$, more energy is transferred to an antisymmetric cross wave than to a symmetric one.

It seems that the analysis of the evolution equations is possible by an approximate analytical WKB method (see *e.g.* [17] p. 558). Such an analysis is currently under development and the results will be presented in a forthcoming paper.

8. Concluding remarks

In contrast to previous research results, the evolution of the cross wave excited by the horizontally oscillating vertical cylinder in water of constant depth is found to be described by two complex, nonlinear partial differential equations with coefficients which depend on a slow radial variable, both in the linear and in the nonlinear terms. These equations differ from the cubic Schrödinger equation which governs progressive cross waves in a rectangular channel (see [13]) and differ from the evolution equations derived in [15] for an axisymmetric wavemaker. The dependence of the linear part on the slow radial variable is a very important feature playing a crucial role in the stability analysis of the forced wave. If the coefficients of the linearized evolution equations were independent of the radial variable, the forced wave would be stable (under the presence of viscous dissipation) and the excitation of the cross wave would be impossible (at least for small perturbations).

It has been shown that, due to the specific geometry of the excitation, energy transfer from the wavemaker to the cross wave occurs through higher-order (quartic) interactions. It determines the scaling of the slow variables, which is different in comparison to previous studies. The evolution process is slower and the radial variable is stretched out.

The functional desribing the energy transfer is quadratic both in the forced wave and in the cross wave and comprises the integrals over the free surface, the wavemaker and over the wavemaker-water line. Therefore, the solution to the second-order problems is necessary for the derivation of the evolution equations. The required solutions for second-order wave components have been determined in the present work through integral transforms.

The derivation of the evolution equations is based on the assumptions that the amplitudes of the forced wave and the cross wave are of the same order. Moreover, two harmonic azimuthal wave components have been included in the trial solution for a linear approximation to a cross wave. These assumptions are in accordance with the results of our experimental studies on various models of offshore structures.

Appendix

A. Calculation of P_s and P_c

The detailed analysis shows that P_s and P_c result from the contributions of the integral over the undisturbed free surface (P^F) and of the line integral over the intersection of the cylinder with the free surface (P^I)

$$P_s = P_s^F + P_s^I, \qquad P_c = P_c^F + P_c^I. \tag{82}$$

The contribution of the integral over the cylinder surface is equal to zero.

The components P_s^F and P_c^F are equal and given by

$$\begin{aligned} P_s^F \;=\; P_c^F = \frac{\pi}{4}\mathfrak{Re}\Bigg\{ i\int_{r_1}^{\infty} \sum_{m=1,3}\ \sum_{\substack{\alpha=(n-1),\\ (n+1)}} \Bigg[\Phi_0^* F_n \left(\left[8(-1)^{\frac{m+1}{2}} + 5 + \frac{\alpha n}{r^2}\right] Z_\alpha^m \right.\\ \left. + \frac{\alpha n}{r^2}\Phi_\alpha^m + 2(m-1)\Phi_{\alpha,z}^m + \Phi_{\alpha,zz}^m\right) + \Phi_{0,r}^* F_n(\Phi_{\alpha,r}^m + 4\Phi_{\alpha,z}^m) \\ + \Phi_0^* F_{n,r}(\Phi_{\alpha,r}^m + Z_{\alpha,r}^m) + \Phi_{0,r}^* F_{n,r} Z_\alpha^m + O(r^{-5/2})\Bigg]_{|z=0} r\,\mathrm{d}r \Bigg\}. \end{aligned} \tag{83}$$

We may calculate the functions Φ_α^m and Z_α^m from (56) and (57) by taking the contribution of the pole $\mu = m^2$ and the neighbourhood of $\mu = 0$, and applying an asymptotic approximation to the Hankel functions for large r and r_1

$$\begin{aligned} \begin{bmatrix} \Phi_\alpha^m \\ Z_\alpha^m \end{bmatrix} \sim -\frac{1}{8}\begin{bmatrix} (-1)^{\frac{m+1}{2}}\, \mathrm{e}^{m^2 z} \\ -m \end{bmatrix} \left(\frac{2\pi m^2}{r}\right)^{1/2} \mathrm{e}^{i[m^2(r-r_1)]} \\ \cdot\left[(2m^4 - 50)m\mathcal{K}_\alpha^{(01)}(m^2) - \mathcal{M}_\alpha(m^2)\right] \\ + \frac{i(-1)^{\frac{m+1}{2}}}{16m\, r}\begin{bmatrix} 50 \\ -\dfrac{45}{m} \end{bmatrix}\left(\frac{2r_1}{\pi}\right)^{1/2}\left[\mathrm{e}^{i[5(r-r_1)-n\frac{\pi}{2}]} - \mathrm{e}^{i[3(r-r_1)+n\frac{\pi}{2}]}\right] + O(r^{-3/2}). \end{aligned} \tag{84}$$

The components P_s^I and P_c^I are given by

$$P_s^I = \frac{\pi}{16} F_n^2(r_1) r_1 \left[6\,\mathfrak{Re}\{\Phi_0\} \left(\frac{n^2}{r_1^2} - 5 \right) - \mathfrak{Im}\{\Phi_0\} \left(\frac{3n^2}{r_1^2} - 1 \right) \right]_{|z=0,\, r=r_1},$$

$$P_c^I = \frac{\pi}{16} F_n^2(r_1) r_1 \left[2\,\mathfrak{Re}\{\Phi_0\} \left(\frac{n^2}{r_1^2} - 5 \right) - \mathfrak{Im}\{\Phi_0\} \left(\frac{n^2}{r_1^2} - 3 \right) \right]_{|z=0,\, r=r_1}. \tag{85}$$

The use of an asymptotic approximation to Φ_0 for large r and r_1 leads eventually to very simple expressions

$$P_s^I = \frac{\pi}{16} F_n^2(r_1) r_1 \left(\frac{3n^2}{r_1^2} - 1 \right), \qquad P_c^I = \frac{\pi}{16} F_n^2(r_1) r_1 \left(\frac{n^2}{r_1^2} - 3 \right) \tag{86}$$

from which follows that $P_s^I > P_c^I$ for any n and r_1. This relationship implies immediately $P_s > P_c$ and we may deduce that it is generally easier to excite an antisymmetric (with respect to the vertical plane of excitation) cross wave than a symmetric one.

The integrand in $\mathcal{K}_\alpha^{(01)}(m^2)$ comprises a triple product of Hankel functions $(O(r^{-1/2}))$ that is highly oscillatory (in r). Its improper integral can be calculated, either via Fresnel integrals after an asymptotic approximation to the integrand has been applied, or directly in the semi-analytical way proposed by Kim and Yue [18] or by Chau and Eatock Taylor [19].

B. Evaluation of Q_1, Q_2 and Q_3

$$\begin{aligned} Q_1 = \frac{\pi}{8} \mathfrak{Re} \Bigg\{ \int_{r_1}^{\infty} \{ & F_n^2(r)(4I_{0,z} + 2I_{2n,z} + 2N_0 + N_{2n}) - F_{n,r}^2(r)(2N_0 + N_{2n}) \\ & -2F_n(r)F_{n,r}(r)(2I_{0,r} + I_{2n,r}) - \frac{n^2}{r^2} F_n^2(r)(4I_{2n} + 2N_0 - N_{2n}) \\ & + \frac{n^2}{r^2} F_n^2(r)[10F_n^2(r) - 2F_{n,r}^2(r) - 3\frac{n^2}{r^2} F_n^2(r)]\} r \,\mathrm{d}r \Bigg\}, \end{aligned}$$

$$\begin{aligned} Q_2 = \frac{\pi}{8} \mathfrak{Re} \Bigg\{ \int_{r_1}^{\infty} \Bigg\{ & F_n^2(r)(4I_{0,z} - 2I_{2n,z} + 2N_0 - N_{2n}) - F_{n,r}^2(r)(2N_0 - N_{2n}) \\ & -2F_n(r)F_{n,r}(r)(2I_{0,r} - I_{2n,r}) + \frac{n^2}{r^2} F_n^2(r)(4I_{2n} - 2N_0 - N_{2n}) \\ & - \frac{n^2}{r^2} F_n^2(r) \left[10F_n^2(r) - 2F_{n,r}^2(r) + \frac{n^2}{r^2} F_n^2(r) \right] \Bigg\} r \,\mathrm{d}r \Bigg\}, \end{aligned}$$

$$Q_3 = \pi \int_{r_1}^{\infty} \frac{n^2}{r^4} F_n^4(r) r \,\mathrm{d}r,$$

where

$$I_\alpha = \lim_{z \to 0^-} \int_0^{\infty} \mathrm{e}^{\mu z} (\mu^2 - 4) \mathcal{K}_\alpha^{(11)}(\mu) F_\alpha(\mu r) \frac{\mu \,\mathrm{d}\mu}{\mu - 4}, \tag{87}$$

$$I_{\alpha,z} = \lim_{z\to 0^-} \int_0^\infty e^{\mu z}(\mu^2-4)\mathcal{K}_\alpha^{(11)}(\mu)F_\alpha(\mu r)\frac{\mu^2\,d\mu}{\mu-4}, \tag{88}$$

$$I_{\alpha,r} = \lim_{z\to 0^-} \int_0^\infty e^{\mu z}(\mu^2-4)\mathcal{K}_\alpha^{(11)}(\mu)F_{\alpha,r}(\mu r)\frac{\mu\,d\mu}{\mu-4}, \tag{89}$$

$$N_\alpha = \lim_{z\to 0^-} \int_0^\infty e^{\mu z}(2\mu-\mu^2-\tfrac{1}{4}\mu^3)\mathcal{K}_\alpha^{(11)}(\mu)F_\alpha(\mu r)\frac{\mu\,d\mu}{\mu-4}, \tag{90}$$

for $\alpha = 0,\ 2n$.

The integrals (87)–(90) result from the second-order cross wave and can be replaced by their asymptotic approximation for large r and r_1. This approximation is dominated by the contribution of the pole ($\mu = 4$) and is given by

$$\tilde{I}_\alpha \sim 24i\left(\frac{2\pi}{r}\right)^{1/2} e^{4i(r-r_1)}\mathcal{K}_\alpha^{(11)}(4) + O(r^{-1}),$$

$$\tilde{I}_{\alpha,z} \sim 96i\left(\frac{2\pi}{r}\right)^{1/2} e^{4i(r-r_1)}\mathcal{K}_\alpha^{(11)}(4) + O(r^{-3/2}),$$

$$\tilde{I}_{\alpha,r} \sim 96i\left(\frac{2\pi}{r}\right)^{1/2} e^{4i(r-r_1)}\mathcal{K}_\alpha^{(11)}(4) + O(r^{-3/2}),$$

$$\tilde{N}_\alpha \sim -48i\left(\frac{2\pi}{r}\right)^{1/2} e^{4i(r-r_1)}\mathcal{K}_\alpha^{(11)}(4) + O(r^{-3/2}).$$

Consequently, the coefficients Q_1, Q_2 and Q_3 can be simplified to single integrals

$$Q_1 = \frac{\pi}{8}\Re\mathfrak{e}\left\{\int_{r_1}^\infty \{\tilde{F}_n^2(r)(4\tilde{I}_{0,z}+2\tilde{I}_{2n,z}+2\tilde{N}_0+\tilde{N}_{2n}) - 2\tilde{F}_n(r)\tilde{F}_{n,r}(r)(2\tilde{I}_{0,r}+\tilde{I}_{2n,r}) \right.$$
$$\left. - \tilde{F}_{n,r}^2(r)(2\tilde{N}_0+\tilde{N}_{2n}) + O(r^{-5/2})\}r\,dr\right\}, \tag{91}$$

$$Q_2 = \frac{\pi}{8}\Re\mathfrak{e}\left\{\int_{r_1}^\infty \{\tilde{F}_n^2(r)(4\tilde{I}_{0,z}-2\tilde{I}_{2n,z}+2\tilde{N}_0-\tilde{N}_{2n}) - 2\tilde{F}_n(r)\tilde{F}_{n,r}(r)(2\tilde{I}_{0,r}-\tilde{I}_{2n,r}) \right.$$
$$\left. - \tilde{F}_{n,r}^2(r)(2\tilde{N}_0-\tilde{N}_{2n}) + O(r^{-5/2})\}r\,dr\right\}, \tag{92}$$

$$Q_3 = \pi\int_{r_1}^\infty \{O(r^{-5})\}\,dr \approx 0. \tag{93}$$

Here, $\tilde{F}_n$ and $\tilde{F}_{n,r}$ denote asymptotic approximations to F_n and $F_{n,r}$, respectively.

The integrals in (91)–(93) as well as in $\mathcal{K}_\alpha^{(11)}$ have highly oscillatory (in r) integrands that are $O(r^{-1/2})$. They can again be calculated with sufficient accuracy in the semi-analytical way presented in the papers mentioned in Appendix A.

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